Problem Set 6 - Solution - LV 141.A55 QISS

1. **SQUID with unequal junctions**

The total current of the asymmetric SQUID is given by the Josephson currents of the two junctions

$$I_{\text{tot}} = I_l \sin(\gamma_l) + I_r \sin(\gamma_r) \tag{1}$$

In general γ_l and γ_r can assume any value between 0 and 2π. However, the difference is fixed by the magnetic flux penetrating the SQUID loop

$$\gamma_l - \gamma_r = 2\pi \frac{\phi}{\phi_0} \tag{2}$$

In order to find the maximal current possible, we use a geometric argument: You can interpret equation (1) as the y-component of a two-dimensional vector sum:

$$\vec{I}_\Sigma = \vec{I}_l + \vec{I}_r$$

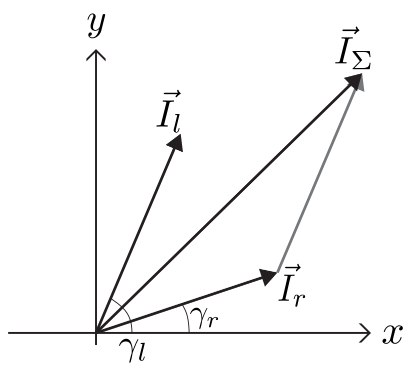

Figure 1: Asymmetric SQUID

The angle between the two vectors is fixed by equation (2). The super current through the two junctions is given by the y-component of the vector $\vec{I}_\Sigma$. The maximal supercurrent is reached when the vector is aligned with the y-axis and given by the absolute value of $|\vec{I}_\Sigma|$.

$$I_{\text{tot,max}} = \sqrt{I_l^2 + I_r^2 + 2 I_l I_r \cos\left(2\pi \frac{\phi}{\phi_0}\right)}$$

Note that the highest supercurrent is reached at integer flux quanta $\phi = n\phi_0$ with $I_{\text{tot,max}} = I_l + I_r$ and the smallest supercurrent at half integer flux quanta $\phi = (n + \frac{1}{2})\phi_0$ with $I_{\text{tot,max}} = |I_l - I_r|$.

2. **RCSJ model - pendulum**

The torque on the pendulum is $\vec{T} = \vec{r} \times \vec{F}$

$$T = -\sin\varphi lmg$$

The angular momentum is $\vec{L} = \vec{l} \times \vec{v}$

$$L = ml^2\omega = ml^2\dot{\varphi}$$

Furthermore we assume an external torque T_{ext} and an angular friction $T_{\text{fr}} = -\theta_{\text{fr}}\omega$.

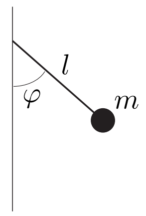

Figure 2: Pendulum

The change of angular momentum L is given by the total torque $\frac{d}{dt}L = T_{\text{tot}}$.

$$T + T_{\text{ext}} + T_{\text{fr}} = \frac{d}{dt}L$$

which leads to a second order differential equation for the angle φ

$$T_{\text{ext}} = ml^2\ddot{\varphi} + \theta_{\text{fr}}\dot{\varphi} + mlg\sin(\varphi)$$

Comparison with the RCSJ model

$$I_{\text{ext}} = C\frac{\phi_0}{2\pi}\ddot{\delta} + \frac{1}{R}\frac{\phi_0}{2\pi}\dot{\delta} + I_0\sin(\delta)$$

The moment of intentia ml^2 corresponds to the capacitance C. The angular friction corresponds to the inverse resistance $1/R$ and the torque due to gravity mlg corresponds to the Josephson current I_0.

3. **RCSJ model - overdamped** For a strongly overdamped, the capacitance of the junction can be neglected and we end up with first order differential equation:

$$I = \frac{1}{R}\frac{\phi_0}{2\pi}\dot{\delta} + I_0\sin(\delta)$$

For a bias current smaller than the critical current I_0 the solution is static (i.e. $\dot{\delta} = 0$) and the voltage zero.

Above the critical current, the solution is complicated function but periodic. Since we are interested only in the average voltage, we calculate the period T of the function

$$\int_0^{2\pi} \frac{d\delta}{\frac{I}{I_0} - \sin(\delta)} = \int_0^T dt\,\frac{2\pi I_0}{\phi_0}R$$

$$\frac{2\pi}{\sqrt{(\frac{I}{I_0})^2 - 1}} = 2\pi\frac{I_0}{\phi_0}RT$$

The average voltage can be calculated with the second Josephson equation

$$V = \frac{\phi_0}{2\pi}\frac{2\pi}{T} = R\sqrt{I^2 - I_0^2}$$